Isometries on the Bloch Space

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Disclaimer

The views and opinions expressed in this talk are those of the speaker and not necessarily those of any other researcher in the field of Function-Theoretic Operator Theory.

Function-Theoretic Operator Theory

Overall Goals of the Field

The overall goal of the field of Function-Theoretic Operator Theory is the study of the interplay between function theory and operator theory. This interplay is most easily seen through the study of operators with symbol, and how the function-theoretic properties of the symbol are linked to the operator-theoretic properties of the operator.

Motivating Example from Linear Algebra

Let A be an $n \times n$ matrix with real coefficients. We define the operator $L_A : \mathbb{R}^n \to \mathbb{R}^n$ by

$$L_A(u) = Au, \quad u \in \mathbb{R}^n.$$

The one-parameter family of operators $\{L_A : A \in M_n(\mathbb{R})\}$ is called an operator with symbol, specifically a left-multiplication operator with symbol A.

Typical Questions Asked of an Operator with Symbol

For a linear operator with symbol $T_f : X \to Y$ between Banach spaces:

1. For what symbol f is T_f a bounded operator?
2. What is the norm of T_f?
3. For what symbol f is T_f an isometry from X to X?
4. What is the spectrum $\sigma(T_f)$?
5. What is the adjoint T_f^*?
6. For what symbol f is T_f compact, normal, subnormal, etc.?

On many spaces, questions 1 and 2 are difficult to answer. Without them, it is difficult to study T_f in any meaningful way.

Types of Operators with Symbol Studied

Typical examples of operators with symbol are:

1. Multiplication Operators:
$$M_\psi(f) = \psi f.$$
2. Composition Operators:
$$C_\varphi(f) = f \circ \varphi.$$
3. Weighted Composition Operators:
$$W_{\psi,\varphi}(f) = \psi(f \circ \varphi).$$

Outline

The Isometry Problem

Isometry Problem

Given a Banach space (of analytic functions) X, what are the isometries on X? That is, what are the linear operators $T : X \to X$ such that

$$||Tx|| = ||x||,$$

for all $x \in X$?

The Isometry Problem was first investigated by Banach in 1932, when he proved that the surjective isometries on $C(Q)$, the space of continuous real-valued functions on a compact metric space Q, are of the form

$$Tf = \psi(f \circ \varphi),$$

where $|\psi| \equiv 1$ and φ is a homeomorphism of Q onto itself.

Unilateral Shift on ℓ^2

$$\ell^2 = \left\{(a_1, a_2, a_3, \dots) : a_n \in \mathbb{C} \text{ and } \sum_n |a_n|^2 < \infty\right\}.$$

$$||(a_1, a_2, a_3, \dots)||_2 = \left(\sum_n |a_n|^2\right)^{1/2}.$$

The unilateral shift operator on ℓ^2 is defined as

$$T(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

T is an isometry on ℓ^2.

Isometries on Classical Spaces

Isometries on H^p

For $0 < p < \infty$, the Hardy Space H^p consists of analytic functions f on the open unit disk $\mathbb{D}$ such that

$$||f||_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p \, d\theta < \infty.$$

[Forelli, 1964] For $p \neq 2$, T is an isometry on H^p if and only if

$$Tf = \psi(f \circ \varphi),$$

for some ψ and φ.

Isometries on Classical Spaces

Isometries on A^p

For $0 < p < \infty$, the Bergman Space A^p consists of analytic functions f on $\mathbb{D}$ such that

$$||f||_p = \left(\int_{\mathbb{D}} |f(z)|^p \ dA\right)^{1/p} < \infty.$$

[Kolaski, 1982] Let $0 < p < \infty$, $p \neq 2$. Then $T : A^p \to A^p$ is a surjective linear isometry if and only if T has the form

$$Tf = \lambda(\varphi')^{2/p}(f \circ \varphi)$$

where φ is an automorphism of $\mathbb{D}$ and λ is a constant of modulus 1.

Introduction to the Bloch Space

Definition

An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be Bloch provided

$$\beta_f := \sup_{z\in\mathbb{D}} (1 - |z|^2) \left| f'(z) \right| < \infty.$$

The Bloch space, defined as $\mathcal{B} = \{f \in H(\mathbb{D}) : \beta_f < \infty\}$, is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f.$$

Schwarz-Pick Lemma

Let $f : \mathbb{D} \to \overline{\mathbb{D}}$ be analytic. Then for $z \in \mathbb{D}$,

$$(1 - |z|^2) \left| f'(z) \right| \leq 1 - |f(z)|^2 .$$

If $f(z)$ is a conformal automorphism of $\mathbb{D}$, then equality holds; otherwise the inequality is strict for all $z \in \mathbb{D}$.

Important Consequences of the Schwarz-Pick Lemma

Proposition

For all $f \in H(\mathbb{D})$, $\beta_f \leq ||f||_\infty$.

Corollary

$H^\infty(\mathbb{D})$ *is contained in* $\mathcal{B}$.

Proposition

For $f \in \mathcal{B}$ *and* φ *analytic from* $\mathbb{D}$ *into* $\mathbb{D}$, *then*

$$\beta_{f \circ \varphi} \leq \beta_f.$$

Moreover, if φ *is a conformal automorphism of* $\mathbb{D}$, *then equality holds;* $\mathcal{B}$ *is Möbius Invariant.*

Examples of Bloch Functions

1. Polynomials
2. Bounded analytic functions
   1. For $a \in \mathbb{D}$ and $|\lambda| = 1$, the Möbius Transformations of $\mathbb{D}$ onto $\mathbb{D}$ are
   $$\varphi_a(z) = \lambda \frac{a - z}{1 - \overline{a}z}.$$
   2. Let $\{a_n\}$ be a sequence in $\mathbb{D}$ satisfying the Blaschke condition of $\sum_n (1 - |a_n|) < \infty$. We define the Blaschke product as
   $$B(z) = z^m \prod_{a_k \neq 0} \left(\frac{\overline{a_k}}{|a_k|} \frac{a_k - z}{1 - \overline{a_k} z} \right),$$
   where $m \in \mathbb{Z}_+$ is the number of times 0 appears in the sequence.
3. Let $f(z) = \operatorname{Log}\left(\frac{1+z}{1-z}\right)$.

Isometries on the Bloch Space

Definition

Let $\mathcal{B}_*$ denote the set of Bloch functions that fix the origin, that is

$$\mathcal{B}_* = \{f \in \mathcal{B} : f(0) = 0\}.$$

Theorem [Cima & Wogen, 1980]

Let $T : \mathcal{B}_* \to \mathcal{B}_*$ be an onto isometry. Then there exists a conformal automorphism φ of $\mathbb{D}$ and a $\lambda \in \partial\mathbb{D}$ so that

$$T(f) = \lambda(f \circ \varphi - f(\varphi(0))),$$

for all $f \in \mathcal{B}_*$.

Such an operator is called a compression of the composition operator C_φ.

Trend in the Study of Isometries on Function Spaces

The current trend in the study of isometries on spaces of analytic functions is to study the isometries of a particular class of operators with symbol T_f.

Family Recipe for Research

(1) Establish conditions on f for which T_f is a bounded operator.

(2) Establish upper and lower estimates on $||T_f||$.

(3) Using the estimates, determine a class of symbols which induce an isometric operator.

(4) Determine a complete characterization of the symbols of isometric operators.

Isometric Composition Operators

Boundedness

As a consequence of the Schwarz-Pick lemma, any analytic self-map of $\mathbb{D}$ induces a bounded composition operator C_φ on the Bloch space.

Norm Estimates [Xiong, 2004]

If φ is any analytic self-map of $\mathbb{D}$, then

$$\max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right\} \leq ||C_\varphi|| \leq \max\left\{1, \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau_\varphi^\infty\right\}$$

where

$$\tau_\varphi^\infty = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2}\left|\varphi'(z)\right|.$$

Isometric Composition Operators

Symbols which induce isometry [Xiong, 2004]

If $\varphi(0) = 0$ and $\beta_\varphi = 1$, then C_φ is an isometry.

Thus rotations $\varphi(z) = \lambda z$ for $|\lambda| = 1$ induce isometric C_φ.

Xiong ended the paper asking the question "Are there any non-rotation symbols φ that induce an isometric C_φ?"

Characterization [Colonna, 2005]

C_φ is an isometry on the Bloch space if and only if $\varphi(0) = 0$ and $\beta_\varphi = 1$. Moreover, φ is a rotation or $\varphi = gB$ where $g : \mathbb{D} \to \overline{\mathbb{D}}$ is a non-vanishing analytic function and B is a Blaschke product whose zeros form an infinite sequence $\{z_n\}$ containing 0 and an infinite subsequence $\{z_{n_j}\}$ such that $\left|g(z_{n_j})\right| \to 1$ and

$$\lim_{j\to\infty} \prod_{k\neq n_j} \left| \frac{z_{n_j} - z_k}{1 - \overline{z_{n_j}} z_k} \right| = 1.$$

Isometric Multiplication Operators

Boundedness [Brown and Shields, 1991]

M_ψ is bounded on the Bloch space if and only if $\psi \in H^\infty(\mathbb{D})$ and

$$\left|\psi'(z)\right| = O\left(\frac{1}{(1-|z|)\log\frac{1}{1-|z|}}\right).$$

Norm Estimates [A. and Colonna, 2008]

If M_ψ is bounded on the Bloch space, then

$$\max\left\{||\psi||_{\mathcal{B}}, ||\psi||_\infty\right\} \leq ||M_\psi|| \leq \max\left\{||\psi||_{\mathcal{B}}, ||\psi||_\infty + \sigma_\psi^\infty\right\}$$

where

$$\sigma_\psi^\infty = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2)\left|\psi'(z)\right|\log\frac{1+|z|}{1-|z|}.$$

Isometric Multiplication Operators

Symbols which induce isometry

Let $\psi(z) = \lambda$ where $|\lambda| = 1$, then M_ψ is an isometry on the Bloch space.

$$||M_\psi f||_{\mathcal{B}} = ||\psi f||_{\mathcal{B}} = |\psi(0)|\,|f(0)| + \beta_{\psi f} = ||f||_{\mathcal{B}}\,.$$

Characterization [A. and Colonna, 2008]

M_ψ is an isometry on the Bloch space if and only if ψ is a constant function of modulus 1.

Isometric Weighted Composition Operators

Boundedness [Ohno and Zhou, 2001]

$W_{\psi,\varphi}$ is a bounded operator on the Bloch space if and only if:

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\left|\psi'(z)\right| \log \frac{2}{1-|\varphi(z)|^2} < \infty,$$

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2} |\psi(z)| \left|\varphi'(z)\right| < \infty.$$

Problematic Example

The following induce a bounded weighted composition operator on the Bloch space:

$$\psi(z) = \log \frac{2}{1-z},$$

$$\varphi(z) = \frac{1-z}{2}.$$

Isometric Weighted Composition Operators

Norm Estimates [A. and Colonna, 2008]

If $W_{\psi,\varphi}$ is bounded on the Bloch space, then

$$\max\left\{ ||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right\} \leq ||W_{\psi,\varphi}|| \leq$$
$$\max\left\{ ||\psi||_{\mathcal{B}}, \frac{1}{2}|\psi(0)| \log \frac{1+|\varphi(0)|}{1-|\varphi(0)|} + \tau^{\infty}_{\psi,\varphi} + \sigma^{\infty}_{\psi,\varphi} \right\}$$

where

$$\tau^{\infty}_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi(z)|^2} |\psi(z)| \left|\varphi'(z)\right|, \text{ and}$$
$$\sigma^{\infty}_{\psi,\varphi} = \sup_{z\in\mathbb{D}} \frac{1}{2}(1-|z|^2) \left|\psi'(z)\right| \log \frac{1+|\varphi(z)|}{1-|\varphi(z)|},$$

Isometric Weighted Composition Operators

Symbols which induce isometry

If ψ induces an isometric multiplication operator M_ψ and φ induces an isometric composition operator C_φ, then $W_{\psi,\varphi}$ is an isometry on the Bloch space.

$$||W_{\psi,\varphi}f||_{\mathcal{B}} = ||M_\psi(C_\varphi f)||_{\mathcal{B}} = ||C_\varphi f||_{\mathcal{B}} = ||f||_{\mathcal{B}}.$$

Characterization

Further Developments

1. Develop a complete characterization of the isometric $W_{\psi,\varphi}$.
2. Develop characterization of isometries for more classes of operators.
3. Generalize to higher dimensions:
   1. How do you generalize $\mathbb{D}$ in higher dimensions?
      - Unit Ball $\mathbb{B}_n$
      - Unit Polydisk $\mathbb{D}^n$
      - Bounded homogeneous domains
   2. How do you relate the operator and function-theoretic properties with the geometry of the space?

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